The wave function representing the particle is

$$\psi(x,t) = A e^{i/\hbar (P_x x - Et)} \quad \text{---} \quad (2)$$

Diff. twice w.r.t x we get

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{1}{\hbar^2} P_x^2 \psi$$

$$\Rightarrow P_x^2 \psi = -\hbar^2 \frac{\partial^2 \psi}{\partial x^2} \quad \text{---} \quad (3)$$

again, diff. w.r.t t

$$\frac{\partial \psi}{\partial t} = \frac{E \psi}{i\hbar}$$

$$\Rightarrow E\psi = i\hbar \frac{\partial \psi}{\partial t} \quad \text{---} \quad (4)$$

Substituting the values of $P_x^2 \psi$ and $E\psi$ in eqn (1) we get,

$$\boxed{-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi = i\hbar \frac{\partial \psi}{\partial t}}$$

→ This is the schrödinger's time dependent wave eqn for a particle moving in a poth V.

Three dimensional Schrödinger eqn

The wave function is give by

$$\psi(\vec{r}, t) = A e^{-i/\hbar (\vec{P}\cdot\vec{r} - Et)}$$

here, P is the momentum and E the total energy of the particle.

$$E = \frac{P^2}{2m} + V(\vec{r}, t)$$

$$\Rightarrow E\psi = \frac{1}{2m}(P_x^2 + P_y^2 + P_z^2)\psi + V(\vec{r}, t)\psi$$

$$\psi(\vec{r},t) = A e^{i/\hbar (p_x x + p_y y + p_z z - Et)}$$

$\therefore$ Diff w.r.t x we get

$$\frac{\partial \psi}{\partial x} = A\left(\frac{i}{\hbar}\right) p_x e^{i/\hbar (p_x x + p_y y + p_z z - Et)}$$

$$= \frac{i}{\hbar} p_x \psi$$

& $$\frac{\partial^2 \psi}{\partial x^2} = \frac{i}{\hbar} p_x \frac{\partial \psi}{\partial x}$$

$$= \left(\frac{i}{\hbar} p_x\right)\left(\frac{i}{\hbar} p_x\right)\psi$$

$$= -\frac{p_x^2}{\hbar^2}\psi$$

$$\therefore p_x^2 \psi = -\hbar^2 \frac{\partial^2 \psi}{\partial x^2}$$

Similarly $p_y^2 \psi = -\hbar^2 \frac{\partial^2 \psi}{\partial y^2}$

& $p_z^2 \psi = -\hbar^2 \frac{\partial^2 \psi}{\partial z^2}$

& Diff. w.r.t t we get

$$\frac{\partial \psi}{\partial t} = A\left(\frac{i}{\hbar}\right)(-E) e^{i/\hbar (p_x x + p_y y + p_z z - Et)}$$

$$= -\left(\frac{i}{\hbar}\right) E \psi$$

$$\therefore E\psi = -\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = i\hbar \frac{\partial \psi}{\partial t}$$

$$\therefore -\frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right) + V(\vec{r},t)\psi = i\hbar \frac{\partial \psi}{\partial t}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\nabla^2 \psi + V(\vec{r},t)\psi = i\hbar \frac{\partial \psi}{\partial t}$$

here, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\Rightarrow \hat{H}\psi = E\psi$$

here, $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r},t) \rightarrow$ 3D Hamiltonian operator

3-D schrödinger eqn for a free particle:

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial\psi}{\partial t}$$

Time-dependent and time independent part of ~~the~~ three dimensional schrödinger wave eqn:-

If $u(\vec{r})$ and $f(t)$ be the time independent and time dependent part of the wave function $\psi(\vec{r},t)$ Then,

$$\psi(\vec{r},t) = u(\vec{r})f(t)$$

$$-\frac{\hbar^2}{2m}f\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right] + Vuf = i\hbar u\frac{\partial f}{\partial t}$$

Dividing both sides by uf

$$-\frac{\hbar^2}{2m}\frac{1}{u}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right] + V = i\hbar\frac{1}{f}\frac{\partial f}{\partial t}$$

The left side of the above eqn is a function of space coordinates whereas the right side is a function of t. There for, each side is equal to a constant E.

$$\therefore -\frac{\hbar^2}{2m}\frac{1}{u}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right] + V = i\hbar\frac{1}{f}\frac{\partial f}{\partial t} = E$$

Time dependent part:-

$$i\hbar\frac{1}{f}\frac{\partial f}{\partial t} = E$$

$\Rightarrow \frac{df}{f} = \frac{1}{i\hbar} E\,dt$

$= -\frac{i}{\hbar} E\,dt$

$\therefore \boxed{f = c\,e^{-\frac{iE}{\hbar}t}}$ where, c is const.

Time independent part of

$$-\frac{\hbar^2}{2m}\frac{1}{u}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]+V=E$$

$$\Rightarrow -\frac{\hbar^2}{2m}\nabla^2 u + Vu = Eu$$

$$\Rightarrow \left[-\frac{\hbar^2}{2m}\nabla^2 + V\right]u = Eu$$

$$\Rightarrow \boxed{\nabla^2 u + \frac{2m}{\hbar^2}(E-V)\,u = 0}$$

Normalisation of the wave function:-

The Schrödinger eqn is linear and homogeneous in ψ and its derivatives $\left(\frac{\partial^2\psi}{\partial x^2}, \frac{\partial\psi}{\partial t}\right)$.

That is why if any solution of it be multiplied by a constant then the resulting wave function will also be a solⁿ. To avoid this arbitrariness it is necessary to impose normalization condition.

Let $\psi_1(r,t)$ be a solⁿ of the schrödinger eqn $\left[\frac{\hbar^2}{2m}\nabla^2+V\right]u = Eu$ and

$$\int_{-\infty}^{+\infty} |\psi_1(\vec{r},t)|^2\, d\tau = N^2$$

where, $|\psi_1(\vec{r},t)|^2 = \psi_1^*(\vec{r},t)\,\psi_1(\vec{r},t)$

N^2 is therefore a real number and is called the norm $\psi(\vec{r},t)$

Let, there be another wave function $\psi(\vec{r},t)$

$$\therefore \psi(\vec{r},t) = \frac{1}{N} \psi_1(\vec{r},t)$$

the we get,

$$\int_{-\infty}^{+\infty} N^2 |\psi(\vec{r},t)|^2 d\tau = N^2$$

$$\Rightarrow \boxed{\int_{-\infty}^{+\infty} |\psi(\vec{r},t)|^2 d\tau = 1}$$

Orthogonality of wave function:-

$$\int_\tau \psi_m^*(\vec{r})\, \psi_n(\vec{r})\, d\tau = 0$$

The condition for ~~orthogo~~ orthonormality $\psi_m(\vec{r})$ and $\psi_n(\vec{r})$ is

$$\int_\tau \psi_m^*(\vec{r})\, \psi_n(\vec{r})\, d\tau = \delta_{mn}$$

δ_{mn} is the Kronecker delta ~~&~~ where,

$$\delta_{mn} = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}$$